NONLINEAR TRANSFORMATION OF LANGMUIR WAVE BEAM IN ISOTHERMAL PLASMA

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We examine the nonlinear mechanisms which lead to the evolution of a given Langmuir wave beam. It is shown that in the large wavenumber region ($v_i/3v_e \ll k\lambda_e \ll$ as a result of induced scattering by ions there is initially a rapid isotropization of the original wave beam and then a slow spectral transfer of the resulting isotropic spectrum in the direction of smaller wavenumbers, which is described by a differential equation. It is shown that in contrast with the region of small $k\lambda_e \ll v_i/3v_e$, where the four-plasmon processes frequently dominate over induced scattering by ions, in the region $k\lambda_e > v_i/3v_e$ four-plasmon scattering gives only small corrections to the kinetic equation for the plasmons. In this connection the real quasistationary turbulent Langmuir wave spectra for $k\lambda_e > v_i/3v_e$ are determined basically by induced ion scattering and not by the four-plasmon processes, as was previously thought.

We examine the evolution of a Langmuir wave "beam" which is given at the initial time. By beam we mean that distribution of plasmons in k-space in which all the plasmons have a wave vector near $\mathbf{k}_0$ in a vicinity of dimension a, where $a \ll k_0$.

The possible nonlinear processes which could determine the nature of the spectral evolution in an isothermal plasma ($T_e = T_i$) include: nonlinear scattering by plasma ions and electrons (see, for example, [1, 2]) and the four-plasmon scattering processes, when two Langmuir plasmons scatter into two others $l + l' \rightleftharpoons l'' + l'''$ [5, 7]. These nonlinear processes lead to isotropization of the wave beam.

To clarify the evolution pattern of the narrow ($\Delta k < k$) isotropic spectrum, from the general expressions for the competing processes of 4-plasmon scattering and scattering by ions in the region $k\lambda_e > v_i/3v_e$ we analyzed the formulas describing the so-called relay-style processes, in which there is spectral evolution of isotropic turbulence as a result of the large number of scattering acts with small change of the wavenumber modulus. It was found that such processes in the region $k\lambda_e > v_i/3v_e$ proceed faster than the scattering processes at large Δk. Particularly fast was the process of relay-style induced scattering by ions, first studied in [3] for the particular case $\Delta k \ll k$, which nearly always predominates over the other nonlinear processes for $k\lambda_e > v_i/3v_e$. In this connection we note that the results of [7] have practically no region of applicability.

We note that the problem of Langmuir wave beam evolution is related directly with the studies devoted to explanation of the observed form of the cosmic ray spectrum on the basis of the assumption on high-frequency Langmuir turbulence in the cosmic plasma [8-10].

Apparently, under astrophysical conditions the primary sources of intense plasma turbulence are the regions behind shock wave fronts. In these regions the turbulence is obviously nonisotropic and nonstationary immediately after shock wave passage. Then there is a transition to the quasistationary isotropic states which form as a result of the balance of turbulence generation in one spectral interval and its transfer across the spectrum into the interval of turbulent pulsation absorption [9, 10].

Specifically, the present investigation of Langmuir wave beam evolution makes it possible to conclude that turbulence isotropization is a faster process than spectral transfer, and therefore the examination of three-dimensional isotropic problems is fully justified.

1. Let us examine the evolution of a Langmuir wave beam on the wavenumber interval

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$$\frac{v_i}{3v_e} \ll k\lambda_e \ll 1 \qquad \left(\lambda_e = \frac{v_e}{\omega_0}\right) \tag{1.1}$$

for induced scattering by plasma ions. Here k is the wavenumber, λ_e is the electron Debye radius. The other notations are standard. Further, we examine a plasma which is nearly isothermal ($T_e = T_i$) in which the existence of ion-sound waves is not possible (for $T_e >> T_i$ the primary nonlinear process [11] will be the decay $l > l' + s$). We note that it is the isothermal plasma which is most often encountered under astrophysical conditions.

Nonlinear scattering of Langmuir waves by plasma ions is described by the equation [1, 2]

$$\frac{\partial N(\mathbf{k})}{\partial t} = N(\mathbf{k}) \int w(\mathbf{p}, \mathbf{k}, \mathbf{k}')(\mathbf{k} - \mathbf{k}') \frac{1}{m_i} \frac{\partial f^{(i)}}{\partial \mathbf{v}} N(\mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\mathbf{p}}{(2\pi)^3} \tag{1.2}$$

If we make the natural assumption that the ion velocity distribution is Maxwellian*

$$f^{(i)}(\mathbf{p}) = \frac{n_0 (2\pi)^{3/2}}{m_i^3 v_i^3} \exp\left(-\frac{p^2}{2m_i^2 v_i^2}\right) \tag{1.3}$$

and substitute into (1.2) the expression for the probability

$$w(\mathbf{p}, \mathbf{k}, \mathbf{k}') = \frac{4(2\pi)^3 e^4 (\mathbf{k}\mathbf{k}')^2 \delta(\omega_k - \omega_{k'} - (\mathbf{k} - \mathbf{k}')\mathbf{v})}{m_e^2 \omega_k^4 k^2 k'^2} \times \left[\frac{\partial \varepsilon^l}{\partial \omega}\bigg|_{\omega_k} \frac{\partial \varepsilon^l}{\partial \omega}\bigg|_{\omega_{k'}}\right]^{-1} \left|\frac{\varepsilon_e^l(\omega_k - \omega_{k'}, \mathbf{k} - \mathbf{k}') - 1}{\varepsilon^l(\omega_k - \omega_{k'}, \mathbf{k} - \mathbf{k}')}\right|^2 \tag{1.4}$$

then on the interval (1.1) we obtain

$$\begin{gathered} \frac{\partial N(\mathbf{k})}{\partial t} = N(\mathbf{k}) \frac{3}{8} \frac{\omega_0 T_e}{T_i m_e n_0 v_i} \int \frac{N(\mathbf{k}')\, d\mathbf{k}'}{(2\pi)^{5/2}} \frac{(\mathbf{k}\mathbf{k}')^2}{k^2 k'^2} \frac{k'^2 - k^2}{|\mathbf{k}' - \mathbf{k}|} \exp\left(-\frac{\omega_-^2}{2k_-^2 v_i^2}\right) \\ \omega_- = \omega_k - \omega_{k'} = \frac{3}{2} \frac{v_e^2 (k^2 - k'^2)}{\omega_0}, \quad \mathbf{k}_- = \mathbf{k} - \mathbf{k}' \end{gathered} \tag{1.5}$$

Equation (1.5), defining the spectral transfer by ions, is written with account for the fact that on the considered interval (1.1) with sufficient accuracy [to $(k\lambda_e)^2$] the ratio

$$\frac{\varepsilon_e^l(\omega_-, \mathbf{k}) - 1}{\varepsilon^l(\omega_-, \mathbf{k}_-)} = 1 - \frac{\varepsilon_i^l(\omega_-, \mathbf{k}_-)}{\varepsilon^l(\omega_-, \mathbf{k}_-)} \approx 1$$

It is easy to see from (1.5) that scattering by ions is maximal for

$$\exp[-\omega_-^2 / 2k_-^2 v_e^2] \sim 1, \qquad \text{or} \qquad \omega_- \lesssim k_- v_i \tag{1.6}$$

In accordance with (1.5) the latter condition may be written as

$$\lambda_e |\mathbf{k} + \mathbf{k}'| \cos\alpha \leqslant 2v_i / 3v_e$$

where α is the angle between the vectors $\mathbf{k} + \mathbf{k}'$ and $\mathbf{k} - \mathbf{k}' = \mathbf{k}_-$.

It must be emphasized that evolution of the Langmuir wave beam in k-space depends on the "seeds" in k-space, i.e., in order that the induced scattering process can begin in an element dk of k-space near $\mathbf{k}_0$ it is necessary that the seeding plasmon level exist there.

Let the seeding plasmon level be nonzero near $\mathbf{k}_0$ at a distance of the order of the dimensions of the initial spectrum in k-space, then the process of diffusion of the spectrum near its maximum in k-space takes place. For the considered beam the wave vectors of the interacting waves are close and, consequent-

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* For the Langmuir wave evolution processes studied in the present paper $\Delta T_i << T_i$ always, i.e., we can use (1.3) and neglect the change of the ion distribution function for nonlinear scattering of Langmuir waves by ions. Ions of all the velocities present in the plasma participate in the scattering process (and not just the ions corresponding to individual segments of the distribution function); therefore the result of the nonlinear process will be general heating of the ion gas. This process can be characterized by the temperature change ΔT_i, which is found from the estimate corresponding to the energy conservation law for spectral transfer of the waves by $\Delta k \sim k$

$$n_0 \Delta T_i \sim N^l \Delta\omega k^2 \Delta k / 2\pi^2 \sim (3/2\pi^2)(k\lambda_e)^2 W^l$$

Hence for $T_e \sim T_i$ we obtain

$$\Delta T_i / T_i \approx (3/2\pi^2)\, u\, (k\lambda_e)^2, \quad u = W^l / n T_e$$

since

$$u \ll 1, \quad k\lambda_e \ll 1, \quad T_i^{-1} \Delta T_i \ll 1$$

ly, $|\mathbf{k}+\mathbf{k}'| = 2k_0$. Therefore (1.6) on the interval (1.1), where $k_0\lambda_e \gg v_i/3v_e$, will be satisfied only for $\cos\alpha \ll 1$. This implies that the maximal increment corresponds to scattering in which the change of the scattering wave vector $\mathbf{k}_-$ is nearly perpendicular to $\mathbf{k}_0$, i.e., most probable is isotropization with small change Δk in modulus. However, we note that scattering in general without change of the modulus of $\mathbf{k}$ also does not take place because of the cofactor $(k'^2 - k^2)$ under the integral sign in (1.5).

Thus, if at the initial time a beam of Langmuir waves is given it will become isotropic as a result of nonlinear scattering by ions. In comparison with the isotropization process the spectral transfer in the direction of smaller $|\mathbf{k}|$ takes place far more slowly. We shall now evaluate the characteristic increment of scattering by ions, having in mind the process of Langmuir wave beam diffusion in the sense described above. It is easy to see that the increment of this process (for broadening of the spectrum by a magnitude of the order of its dimensions in k-space) is maximal for

$$\frac{3}{2}\frac{v_e^2}{\omega_0}(k^2 - k'^2) \sim v_i|\mathbf{k} - \mathbf{k}'| \tag{1.7}$$

If the beam is a "cloud" of sufficiently small dimension a in k-space $a\lambda_e \ll v_i/3v_e \ll k\lambda_e$, then in the integral on the right in (1.5) all the $\mathbf{k}'$ from $N(\mathbf{k}')$ yield a contribution

$$\gamma_{(i)}^{(0)} \approx \omega_0 u, \qquad u = W^l / n_0 T_e \tag{1.8}$$

It is obvious that this first fastest diffusion stage passes rapidly. Assuming that the initial spectrum dimension $a > v_i/3v_e\lambda_e$, we find that in accordance with (1.5) spectral wave transfer can occur only in those $\mathbf{k}$ such that $|\mathbf{k}| < |\mathbf{k}'|$ from $N(\mathbf{k}')$. The layer in the initial spectrum cloud which is cut out by the exponential function in (1.5) has a thickness of order

$$\Delta k \sim a v_i / 3k_0\lambda_e v_e \tag{1.9}$$

The estimate (1.9) is easily obtained from (1.7) if we substitute

$$k + k' \sim 2k, \qquad |\mathbf{k} - \mathbf{k}'| \sim a$$

Therefore, of all the scattered plasmons of the given spectrum $N(\mathbf{k}')$ only a fraction of order

$$\Delta k / a = v_i / 3v_e k_0 \lambda_e \tag{1.10}$$

scatters to the point $\mathbf{k}$.

Hence the unknown increment has the order

$$\gamma_{(i)}^{(1)} \approx \omega_0 u v_i / 3v_e k_0 \lambda_e \tag{1.11}$$

Now let the seed level $N(\mathbf{k})$ be sufficient for wave vectors for any angles between $\mathbf{k}$ and $\mathbf{k}_0$ (but in modulus $|\mathbf{k}|$ close to $|\mathbf{k}_0|$), then by analogy with (1.9) we obtain easily

$$\Delta k \sim v_i / 3v_e \lambda_e \tag{1.12}$$

and correspondingly

$$\gamma_{(i)}^{(2)} \approx \omega_0 u v_i / 3v_e a \lambda_e \gg \gamma_{(i)}^{(1)} \qquad (k_0 \gg a) \tag{1.13}$$

In real conditions, obviously, the seed densities of the plasmon number exist in all directions $\mathbf{k}$, and therefore we should assume that, as a rule, the Langmuir wave beam which is given at the initial time will be rapidly scattered (become isotropic) in all directions more or less uniformly with the increment (1.13).

In comparison with this process, scattering through large angles by electrons with $\gamma^{(e)} \sim \omega_0 u\,(k\lambda_e)^3$ is always insignificant.

2. The nonlinear process of four-plasmon scattering of Langmuir waves $l + l' \rightarrow l'' + l'''$ on the same wavenumber interval (1.1) in certain cases can compete with the induced scattering process examined in the previous section.

The increment of four-plasmon scattering of Langmuir waves can be estimated using the equation

$$\gamma^{(4)} = \int \frac{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3}{(2\pi)^9} w_{ll}^{ll}(\mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_3, \mathbf{k}) N(\mathbf{k}_2) N(\mathbf{k}_3) \tag{2.1}$$

where the process probability has the form [10]

$$w_{ll}^{ll}(\mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_3, \mathbf{k}) = \frac{(2\pi)^9 e^4 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})\, \delta(\omega \mathbf{k}_1 + \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3) - \omega(\mathbf{k}))}{2m_e^2 v_e^4 (4\pi)^3 k^2 k_1^2 k_2^2 k_3^2}$$

$$\times \left\{ \frac{\varepsilon_i^l(k_2 - k_3)}{\varepsilon^l(k_2 - k_3)} \mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 + \frac{\varepsilon_i^l(k_1 - k_3)}{\varepsilon^l(k_1 - k_3)} \mathbf{k}\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3 - \lambda_e^2 [\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3(\mathbf{k}_2 - \mathbf{k}_3)^2 + \mathbf{k}_3\mathbf{k}_1\mathbf{k}\mathbf{k}_2(\mathbf{k}_1 - \mathbf{k}_3)^2] \right. \quad (2.2)$$

$$+ \lambda_e^2 \left[\frac{4}{3} \frac{k_1^2 k_2^2 - (\mathbf{k}_1\mathbf{k}_2)^2}{k_+^2} (\mathbf{k}_1\mathbf{k}_2 k_+^2 - \mathbf{k}\mathbf{k}_+\mathbf{k}_3\mathbf{k}_+) + k_+^2 (\mathbf{k}_1\mathbf{k}_2)^2 + \mathbf{k}\mathbf{k}_1\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2^2 + \mathbf{k}\mathbf{k}_2\mathbf{k}_3\mathbf{k}_2\mathbf{k}_1^2 \right.$$

$$\left.\left. - \mathbf{k}_1\mathbf{k}_2 (k_1^2 \mathbf{k}_2\mathbf{k}_+ + k_2^2 \mathbf{k}_1\mathbf{k}_+ + \mathbf{k}_3\mathbf{k}_1\mathbf{k}\mathbf{k}_1 + \mathbf{k}_3\mathbf{k}_2\mathbf{k}\mathbf{k}_2) + \mathbf{k}\mathbf{k}_1\mathbf{k}_3\mathbf{k}_2 \frac{(\mathbf{k} - \mathbf{k}_1)^2 + (\mathbf{k}_2 - \mathbf{k}_3)^2}{2} + \mathbf{k}\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1 \frac{(\mathbf{k}_1 - \mathbf{k}_3)^2 + (\mathbf{k} - \mathbf{k}_2)^2}{2} \right] \right\}$$

where

$$k_+ = k_1 + k_2 = k + k_3, \quad k_2 - k_3 = \{\mathbf{k}_2 - \mathbf{k}_3, \quad \omega_2 - \omega_3\}$$
$$k_1 - k_3 = \{\mathbf{k}_1 - \mathbf{k}_3, \ \omega_1 - \omega_3\}, \ \omega(k) = \omega_0 + {}^3/_2 k^2 v_e^2 \omega_0^{-1}$$

On the interval (1.1) the maximal four-plasmon decay increment is expressed by the equation

$$\gamma_0^{(4)} \approx \frac{\pi}{128} \left(\frac{k}{a}\right)^2 u^2 (k\lambda_e)^{-2} \omega_0 \qquad (2.3)$$

Here (2.3) corresponds only to those disintegrations in which each of the plasmons participating in the process alters the modulus of its wave vector only slightly, i.e.,

$$\varepsilon_i^l(k_-) / \varepsilon^l(k_-) = {}^1/_2$$

in (2.2). For the considered beam this corresponds to scattering across the beam, i.e, isotropization of the initial wave spectrum.

For processes with relatively large changes of the wave vector modulus

$$\frac{\varepsilon_i^l(k_-)}{\varepsilon^l(k_-)} = \frac{m_e}{9m_i} \frac{1}{(k\lambda_e)^2} \quad \text{for} \quad \frac{v_i}{3v_e} \ll k\lambda_e \ll \frac{1}{\sqrt{3}} \left(\frac{m_e}{m_i}\right)^{1/4} \qquad (2.4)$$

$$\frac{\varepsilon_i^l(k_-)}{\varepsilon^l(k_-)} = (k\lambda_e)^2 \quad \text{for} \quad \frac{1}{\sqrt{3}} \left(\frac{m_e}{m_i}\right)^{1/4} \ll k\lambda_e \ll 1 \qquad (2.5)$$

Therefore the increment $\gamma^{(u)}$ equals on the interval (2.4)

$$\gamma_I^{(4)} \sim \gamma_0^{(4)} (m_e / m_i)^2 (3k\lambda_e)^{-4} \qquad (2.6)$$

and on the interval (2.5)

$$\gamma_{II}^{(4)} \sim \gamma_0^{(4)} (k\lambda_e)^4 \qquad (2.7)$$

Thus it is easy to see that the four-plasmon scattering processes with considerable change of the modulus $\Delta \mathbf{k} = |\mathbf{k}_1| - |\mathbf{k}| \sim |\mathbf{k}|$, which yield broadening of the spectrum with respect to the modulus $|\mathbf{k}|$, takes place very slowly in comparison with isotropization.

Let us compare the nonlinear processes of scattering by ions and four-plasmon scattering and find under what conditions on the interval (1.1) the four-plasmon processes dominate over the induced scattering by the plasma ions. The inequality $\gamma_0^{(4)} > \gamma_i^{(2)}$ yields

$$U \gg \frac{128}{\pi} \frac{v_i}{v_e} a\lambda_e \qquad (2.8)$$

As the spectrum broadens (a/k) increases,and thus in accordance with (2.8) only for sufficiently narrow beams and high wave energy can the four-plasmon processes determine the beam evolution. When (2.8) is violated the induced scattering by plasma ions begins to be the primary nonlinear process. Considering the condition for applicability of the random phase approximation

$$\gamma^{(4)} \ll \Delta\omega = 3\lambda_e^2 a k \omega_0$$

we obtain a limitation on u which defines the region of dominance of four-plasmon interaction of waves with random phase

$$U \ll 4 (k\lambda_e)^2 \left(\frac{2a}{k}\right)^{3/2} \qquad (2.9)$$

We find from (2.8) and (2.9) that for the existence of a region of dominance of four-plasmon interactions for waves with random phase it is necessary that

$$a\lambda_e \gg 3.6 v_i / v_e \tag{2.10}$$

This condition on the interval (1.1) is quite severe on the "size" a/k and $k\lambda_e$ of the beam in k-space; therefore we can conclude that for waves with random phase on the interval (1.1) the process of induced scattering by plasma ions is always in practice the primary nonlinear process leading to evolution of the Langmuir wave beam.

3. It has been shown that the evolution of a Langmuir wave beam which is given at the initial time leads to isotropization. As a result the spectrum takes in k-space the form of a spherical layer of thickness a and radius $\sim k_0$.

For the isotropic case and sufficiently smooth spectrum in the region (1.1), (1.5) can be reduced to differential form. With account for the fact that

$$k - k' \ll k, \quad k_- = |\mathbf{k} - \mathbf{k}'| = k\sqrt{2(1-\cos\theta)}$$
$$\cos\theta = \frac{\mathbf{k}\mathbf{k}'}{kk'}$$

therefore

$$\frac{\omega_-}{k_- v_i} = \frac{3}{\sqrt{2}} \frac{v_e}{v_i} \frac{(k-k')}{\sqrt{1-\cos\theta}}$$

Substituting k_- into (1.5), we obtain for the isotropic case

$$\frac{\partial N(k)}{\partial t} = N(k) \frac{3\omega_0 T_e}{8 T_i m_e n_0 v_i} \int \frac{N(k')\, dk' k'^2 \cos^2\theta\, d\cos\theta\, \sqrt{2}(k'-k)}{(2\pi)^{3/2} \sqrt{1-\cos\theta}} \exp^{-\frac{\alpha^2 (k'-k)^2}{2}}$$
$$\alpha = \frac{3}{\sqrt{2}} \frac{v_e}{v_i} (1-\cos\theta)^{-1/2} \tag{3.1}$$

Similarly [2, 3], expanding the turbulent pulsation spectral energy density W(k) into a series in $(k'-k)$ and integrating the right side first with respect to $d\alpha\,(k'-k)$ and then $d\cos\theta$, we obtain* ($T_e = T_i$)

$$\frac{\partial W(k)}{\partial t} = W(k) \frac{\partial W(k)}{\partial k} \frac{\pi\omega_0^3}{27 m_i n_0 v_e^4}, \quad W(k) = \frac{\omega_0 k^2 N(k)}{2\pi^2} \tag{3.2}$$

Equation (3.2) corresponds to the situation in which in each induced scattering act there is only a small change of the frequency (modulus of the wave vector $\Delta k \leqslant v_i / v_e \lambda_e$). Transfer by relatively large Δk can be achieved for a large number of scattering acts, i.e., the process has a "relay-type" nature.

It is easy to see that (3.2) corresponds to spectral transfer in the direction of lower frequencies. The effect of some narrowing of the spectrum [2, 3] is small, since $(k\lambda_e)^2$ is comparable with 1. It is easy to obtain from (3.2) the estimate of the increment of the corresponding process

$$\gamma^{(i)*} \sim \frac{\pi}{27} \omega_0 \frac{m_e}{m_i} u (k\lambda_e)^{-2} (k/\Delta k)^2 \tag{3.3}$$

By analogy with the process of induced scattering by ions, omitting the tedious calculations, we can write in the region (1.1) for the isotropic case and sufficiently smooth spectrum the expression corresponding to relay-type spectral transfer of Langmuir waves for the four-plasmon processes

$$\frac{\partial N(k)}{\partial t} = -\int \frac{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3}{(2\pi)^9} w_{ll}^{ll}(\mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_3, \mathbf{k}) \{N(\mathbf{k}) N(\mathbf{k}_1) N(\mathbf{k}_3)$$
$$+ N(\mathbf{k}) N(\mathbf{k}_2) N(\mathbf{k}_3) - N(\mathbf{k}) N(\mathbf{k}_1) N(\mathbf{k}_2) - N(\mathbf{k}_1) N(\mathbf{k}_2) N(\mathbf{k}_3)\}$$
$$= D \int \left\{ F_1(k_1, k_2) \left(N^2(k) \frac{\partial N(k_2)}{\partial k_2^2} - N^2(k_2) \frac{\partial N(k)}{\partial k^2} \right) + F_2(k_1 k_2) \left(N(k) - N(k_2) \right) \frac{\partial N(k)}{\partial k^2} \frac{\partial N(k_2)}{\partial k_2^2} \right\} dk_2^2 \tag{3.4}$$

Here

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* The result [2, 3] was obtained in the particular case of a narrow isotropic spectrum $\Delta k' \ll k$. However, if in the corresponding equation of [2, 3] we neglect the small terms of order $k^{-1}\Delta k$ retained there in comparison with 1, the results of [2, 3] coincide with (3.2) for $\Delta k \sim k$.

$$D = \frac{\omega_0{}^5\pi}{3v_e{}^2\,(4\pi)^5(nT_e)^2}\left(\frac{4}{3\lambda_e}\sqrt{\frac{m_e}{m_i}}\right)^3\int\limits_0^\infty y^2 f^2(y)\,dy \tag{3.5}$$

$$f(y) = \left(1 - I_+(y)\right) \Big/ \left(1 - I_+(y) + \frac{T_e}{T_i}\right)$$

$$F_1(k, k_2) = \int\limits_{-1}^{1} d\cos\theta\,\frac{k_2 k_1{}^2}{|\mathbf{k}_2 - \mathbf{k}|}\left(\frac{A_1}{k^2} + \frac{A_2}{k_2{}^2}\right), \quad F_1 \sim k \ \text{ for } \ k_2 \sim k$$

$$F_2(k, k_2) = \int\limits_{-1}^{1} d\cos\theta\, A k_\perp{}^3 k_2 / |\mathbf{k}_2 - \mathbf{k}|, \quad F_2 \sim k \ \text{ for } \ k_2 \sim k$$

$$k_\perp = \frac{k k_2 \sin\theta}{|\mathbf{k}_2 - \mathbf{k}|} = \frac{k k_2 \sin\theta}{\sqrt{k_2{}^2 + k^2 - 2 k k_2 \cos\theta}}, \quad k_\parallel = \sqrt{k^2 - k_\perp{}^2}$$

$$k_{2\parallel}{}^2 = k_2{}^2 - k_\perp{}^2, \qquad \cos\theta = \frac{\mathbf{k}\mathbf{k}_2}{k k_2}$$

$$A_1 = -\frac{32}{15}\frac{k_\parallel{}^2}{k^2}\left[1 - \frac{12}{7}k_\perp{}^2\left(\frac{2}{k_2{}^2} + \frac{1}{k^2}\right) + \frac{64}{21}\left(\frac{1}{k_2{}^4} + \frac{2}{k^2 k_2{}^2}\right)k_\perp{}^4 - \frac{1280}{231}\frac{k_\perp{}^6}{k^2 k_2{}^4}\right] \tag{3.6}$$

$$A_2 = -\frac{32}{15}\frac{k_{2\parallel}{}^2}{k^2}\left[1 - \frac{12}{7}k_\perp{}^2\left(\frac{2}{k^2} + \frac{1}{k_2{}^2}\right) + \frac{64}{21}\left(\frac{1}{k^4} + \frac{2}{k^2 k_2{}^2}\right)k_\perp{}^4 - \frac{1280}{231}\frac{k_\perp{}^6}{k_2{}^2 k^4}\right]$$

$$A = \frac{4}{3}\left\{1 - \frac{16}{5}k_\perp{}^2\left(\frac{1}{k^2} + \frac{1}{k_2{}^2}\right) + \frac{96}{35}k_\perp{}^4\left(\frac{1}{k^4} + \frac{1}{k_2{}^4} + \frac{4}{k^2 k_2{}^2}\right) - \frac{1024}{105}k_\perp{}^6\left(\frac{1}{k^2} + \frac{1}{k_2{}^2}\right)\frac{1}{k^2 k_2{}^2} + \frac{2560}{231}\frac{k_\perp{}^6}{k_2{}^2 k^4}\right\}$$

The expression for $I_+(y)$ may be found, for example, in [13]. Strictly speaking, the expression (3.4) is valid only for

$$k\lambda_e < \frac{1}{\sqrt{3}}\left(\frac{m_e}{m_i}\right)^{1/2}$$

If the inequality sign is reversed the spectral transfer does not have a relay-type nature (see the estimate (11) of [10]).

Using (3.4)-(3.6) it is easy to obtain the estimate of the increment for such a process

$$\gamma^{(4)*} \sim \omega_0 u^2\left(\frac{m_e}{m_i}\right)^{3/2}\left(\frac{k}{\Delta k}\right)\frac{1}{4}(3k\lambda_e)^{-5} \tag{3.7}$$

The condition for dominance of induced scattering by ions over the four-plasmon processes in the isotropic case

$$u \ll 36\pi\left(\frac{\Delta k}{k}\right)(k\lambda_e)^3\left(\frac{m_i}{m_e}\right)^{1/2} \ \text{ for } \ \frac{v_i}{3v_e} \ll k\lambda_e \ll 1,\ u \ll 1 \tag{3.8}$$

is nearly always satisfied owing to the large coefficient in the right side of (3.8). Therefore the term corresponding in the complete kinetic equation to the four-plasmon processes should be considered only as a correction term for vanishing of the transfer because of scattering by ions (for example $\partial(k^2 N(k))/\partial k) = 0$.

4. Let us estimate the increments of four-plasmon scattering and scattering by ions for a Langmuir wave beam if

$$k\lambda_e \ll v_i / 3v_e \tag{4.1}$$

The maximal increment of four-plasmon scattering for the process of spectral broadening in k-space by a magnitude of order of the initial spectrum width is defined by the expression

$$\gamma^{(4)} \approx \frac{\pi}{8}\left(\frac{T_e}{T_e + T_i}\right)^2\left(\frac{k}{2a}\right)^2 u^2 (k\lambda_e)^{-2}\omega_0 \tag{4.2}$$

In estimating the increment here we have considered that for $k\lambda_e \ll v_i/3v_e$ in (2.2) the ratio $\varepsilon_i^l(k_-)/\varepsilon_e^l(k_-) \sim T_e/(T_e + T_i)$ (for $v_i/3v_e < k\lambda_e < {}^1/_3\sqrt{m_e/m_i}$, which makes sense holds only for a nonisothermal plasma $T_e \gg T_i$, $\varepsilon_i^l(k_-)/\varepsilon^l(k_-) \approx 1$).

In estimating the increment we considered that $N(k) = 6\pi^2 W^l/\omega_0 a^3$ in a small spherical region of k-space of radius a near k_0 ($k_0 \gg a$). The increment of Langmuir wave scattering by ions in the region (4.1) for spectral transfer in k-space by a magnitude of the order of the initial spectrum width is defined by the expression

$$\gamma^{(i)} \approx \omega_0 \frac{2(T_e/T_i)^{3/2}}{(1+T_e/T_i)^2} k\lambda_e \left(\frac{m_e}{m_i}\right)^{1/2} u \tag{4.3}$$

The conditions for dominance of four-plasmon scattering $\gamma^{(4)} >> \gamma^{(i)}$ can be written in the form

$$u > u_1 = \left(\frac{2a}{k}\right)^2 \frac{16}{\pi} \left(\frac{T_i}{T_e}\right)^{1/2} \left(\frac{m_i}{m_e}\right)^{1/2} (k\lambda_e)^3 \tag{4.4}$$

Now we shall account for the condition of applicability of the random-phase approximation

$$\gamma^{(4)} \ll \Delta\omega = 3\lambda_e^2 2ka\omega_0, \quad \text{or} \quad u \ll u_2 = \left(\frac{2a}{k}\right)^{3/2} \left(\frac{96}{\pi}\right)^{1/2} (k\lambda_e)^2 \frac{T_e+T_i}{2T_e} \tag{4.5}$$

and the condition for insignificance of Coulomb interactions

$$\gamma^{(4)} \gg \gamma_{st} = \omega_0 / n\lambda_e^3, \quad \text{or} \quad u \gg u_3 = \frac{32}{\pi} \frac{2a}{k} k\lambda_e \frac{T_e+T_i}{2T_e} (n\lambda_e^3)^{-1/2} \tag{4.6}$$

The conditions (4.4)-(4.6) define the region of dominance of four-plasmon interaction of waves with random phase over the other processes for

$$u_1 < u < u_2 \tag{4.7}$$

As for the criterion (4.6), usually $u_3 < u_1$.

Considering both (4.5) and (4.6), we obtain the necessary condition for the existence of a region of dominance of four-plasmon processes over collisions for waves with random phase

$$n_0 \ll (a\lambda_e)^2 (k\lambda_e)^2 \left[\frac{9}{16\pi^3} \left(\frac{m_e}{m_i}\right)^2 \left(\frac{T_e}{e^2}\right)^3\right] \tag{4.8}$$

The condition can be considered as a limitation on T_e and n_0 for given $a\lambda_e$ and $k\lambda_e$. This yields, for example, for $T_e \sim$ 1ev, $k\lambda_e = 10^{-2}$, $a/k = 0.1$, and a hydrogen plasma $h << 5 \cdot 10$ cm^{-3}.

An interval of values of u permitted by (4.4) and (4.5) exists for

$$\frac{2a}{k} \ll \frac{m_e}{m_i} (k\lambda_e)^2 \left(\frac{T_e+T_i}{2T_e}\right)^2$$

i.e., practically always for $k\lambda_e < v_i/3v_e$ and $a < k/2$.

Let us take, for example, a hydrogen plasma with $T_e = T_i$, $a = 0.05k_0$ and $k\lambda_e = 0.1\,(m_e/m_i)^{1/2}$. In this case, for dominance of the four-plasmon processes over nonlinear scattering by ions and, at the same time, in order that the random phase approximation still be applicable, it is necessary that the wave energy density u lie in the range.

$$2.5 \cdot 10^{-8} \ll u \ll 10^{-6}$$

We note that by virtue of (1.5) and $k\lambda_e << v_i/3v_e$ the wave-energy density has the upper limit $u << (m_e/m_i)$. In the case of lower energies than those defined by (4.4), $u_3 < u < u_1$ and induced scattering by ions dominates over four-plasmon scattering. We note that there is a qualitative difference in the spectral evolution of the Langmuir wave beam in k-space in the interval $v_i/3v_e << k\lambda_e << 1$ and for $k\lambda_e << v_i/3v_e$. In the latter case because of induced scattering by ions the initial spectrum corresponding to the Langmuir wave beam displaces in k-space in the direction of smaller k and narrows somewhat in modulus at the same time it becomes isotropic (see [5]), which is not the dominant process in this case.

The evolution of isotropic turbulent spectra on the interval $k\lambda_e < v_i/3v_e$ was examined in detail in [3] and the quasistationary isotropic states were examined in [9, 10].

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